STATISTICAL CALCULATION OF THE GEOMETRICAL FACTORS IN RADIANT TRANSFER

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A method is proposed for calculating the generalized geometrical factors in radiant transfer for a system of second-order surfaces bounded by their lines of intersection.

The use of Monte Carlo methods to determine the geometrical factors in radiant transfer has been described by Howell and Perlmutter [5], and the application of those methods to the generalized geometrical factors for a system of plane surfaces has been discussed in [1]. In the present article we give a more general algorithm for determining the generalized geometrical factors (i.e., on the assumption that the emitters are black bodies), permitting the results of the calculations to be used to immediately determine the geometrical factors between all surfaces comprised in a system, even when they have complex shapes and boundaries (specifically, those described by second-order equations). The subsequent determination of the solving generalized geometrical factors does not in itself present any difficulty [2].

The proposed method can be likened to the following physical experiment. Let us consider the determination of the average geometrical factors. Suppose that the investigated system of surfaces is enclosed in a spherical black shell. In thermodynamic equilibrium the emission into the interior of the sphere is isotropic. Let us imagine that the investigated surfaces are perfectly transparent. This supposition does not effect any changes in the radiant energy distribution in the interior. It is now readily apparent that the radiation transferred across each surface is "ideally diffuse" with respect to the latter, i.e., its intensity obeys the Lambert cosine law. Consequently, the emissive source in the system is the inner surface of the sphere. Were it possible to pinpoint random rays, it would then be possible to find the average geometrical factor of the radiation from the i-th to the j-th surface by determining a certain number of rays transmitted across the i-th surface (N_i) and, of that number, the number of rays transmitted across the j-th surface (N_{ij}), using the following relation:

$$\varphi_{ij} = N_{ij}/N_i. \qquad (1)$$

The accuracy of the computation of φ_{ij} increases with the number of rays that are registered.

The proposed Monte Carlo method is based on the mathematical formalization of the foregoing physical experiment. For this stochastic model each elementary experiment entails the following. A ray is drawn from a random point on the surface of the sphere inwardly in a direction whose probability corresponds to the cosine law (Lambert emission law). If the ray intersects the i-th surface, the number N_i is increased by one. If the ray also crosses the j-th surface, N_{ij} is increased by the amount $\exp(-Kl)$, where K is the average absorption coefficient of the medium over the ray path l between the surfaces. If a large number of elementary experiments is performed, a computation according to (1) yields a result close to φ_{ij}.

Before continuing, we show that the emission from the sphere into its interior is isotropic if the emission from the surface of the sphere obeys the Lambert law. Thus, the radiant flux across a plane surface element arbitrarily situated and oriented in the interior of the sphere does not depend on its position, because the geometrical factor from it to the spherical surface is equal to unity. Thus, regardless of the position of the elementary area, it is "intercepted by an equal number" of rays. Each surface element, on the other hand, "emits" the same number of rays, whose distribution with respect to the directions

relative to the given element obeys the Lambert cosine law. The algorithm used to compute the geometrical factors involves the following operations:

1. Designate the equations of the surfaces. Enclose the investigated volume in a sphere S of radius ρ. The radius of the sphere must be made as small as possible so that the investigated volume will fill up the maximum possible fraction of the sphere interior. It is noted that this observance will reduce the number of elementary experiments, because there will be fewer "idle" rays, i.e., rays that do not intersect any of the investigated surfaces.

2. Select a ray at random.

3. Determine the points of intersection, providing they exist, of the ray and a surface in the domain of existence of each surface.

4. Vary N_i and N_{ij} when the ray intersects the i-th and j-th surfaces, respectively. Repeat operations 2, 3, and 4 N times. The higher the computational accuracy required, the larger must be the number N.

5. Determine the generalized geometrical factors from Eq. (1).

1. The equation for a second-order surface

$$\sum_{i,j=1}^{4} A_{ij}x_i x_j = 0 \qquad (x_4 = 1, \ A_{ij} = A_{ji})$$

can be specified by designating the ten coefficients $A_{ij}(j \geq i)$ or the coordinates of ten points of the surface.

In the latter case the unknown coefficients A_{ij} of the approximating surface are found by solving the system of nine linear homogeneous equations

$$\sum_{ij=1}^{4} A_{ij}x_{ih}x_{jh} = 0 \qquad (x_{4h} = 1, \ A_{ij} = A_{ji}), \tag{2}$$

in which k = 1, 2, ..., 9 enumerates the points on the surface with coordinates $(x_{1k}, \ x_{2k}, \ x_{3k})$ in order.

The system (2) contains ten unknown coefficients. Inasmuch as the system is homogeneous, it can be used to find the ratios of all the coefficients to one coefficient picked at random. We choose A_{44} for the reference coefficient, i.e., we put $A_{44} = 1$. For the same surfaces for which in the given coordinate system $A_{44} = 0$ (i.e., the determinant of the system is equal to zero) we find the coefficients A_{ij} of the equation for the surface in the new coordinate system

$$x'_{jk} = x_{jk} + \xi_j,$$

and we then transform them for the old coordinate system:

$$\cdot A_{ij} = A'_{ij}. \qquad A_{44} = 0,$$
$$A_{i4} = -\sum_{j=1}^{4} A_{ij}\xi_j \qquad (i, \ j = 1, \ 2, \ 3). \tag{3}$$

Here the ξ_j are arbitrary numbers. Bearing the foregoing remarks in mind, we choose the sphere S for a cube so that the latter is exactly inscribed by it.

2. The selection of a ray at random is equivalent to the selection of a point at random on the surface of the sphere "ρ" and the selection of a ray direction.

To determine the coordinates of a random point $(x_{11}, \ x_{21}, \ x_{31})$ on the surface of a sphere S of radius ρ with center at the point $(x_{10}, \ x_{20}, \ x_{30})$ it must be taken into account that the density of those points on the surface has to be constant. This means that for a large number of random points their concentration on a surface element dS must be proportional to the value of dS, where

$$dS = \rho^2 \sin \theta d\theta d\beta = -\rho^2 d\cos \theta d\beta.$$

Here β and θ are the longitude and polar distance ($-\pi < \beta < \pi$, $0 < \theta < \pi$, $-1 < \cos \theta < 1$). Hence it is clear that if we choose the coordinates of the random point from the set of values $\xi_1 = \beta/\pi$ and $\cos \theta = \xi_2$ in the interval (1, −1), then for a large number of experiments the points will be uniformly distributed over the surface of the sphere. Thus, $\theta = \cos^{-1} \xi_2$, and the Cartesian coordinates of the point are

$$x_{11} = \rho \sin \theta \cos \beta + x_{10} = \rho \sin (\text{Arccos } \xi_2) \cdot \cos \pi \xi_1 + x_{10}$$
$$= \rho \sqrt{1 - \xi_2^2} \cos \pi \xi_1 + x_{10};$$
$$x_{21} = \rho \sin \theta \sin \beta + x_{20} = \rho \sqrt{1 - \xi_2^2} \sin \pi \xi_1 + x_{20};$$
$$x_{31} = \rho \cos \theta = \rho \xi_2 + x_{30}.$$

The foregoing result can be interpreted as a uniform spatial distribution of rays emitted by the point 0 or as the equiprobability of any ray in any direction. Bearing this remark in mind, we determine the random direction of a ray emanating from a point (x_{11}, x_{21}, x_{31}) in the interior of the sphere. We adopt that point as the origin of a new coordinate system $X_1' X_2' X_3'$ and enclose the point (x_{11}, x_{21}, x_{31}) in a unit sphere. Let the X_3' axis coincide with the normal to dS, and let the X_3 and X_2' axes be mutually perpendicular thereto. Bearing in mind the remark made above, we infer that in order to determine the directions of the random rays, the density of whose traces on the surface of the unit sphere (or in space) must obey the cosine law, the coordinates β' and θ' must be selected at random from the set $\xi_4 = \beta'/\pi$ in the interval $(-1, 1)$ and $\cos^2\theta' = \xi_3$ in the interval $(0, 1)$. Thus, $\theta' = \sin^{-1}\sqrt{\xi_3}$, and the coordinates on the surface of the unit sphere of the point through which the ray passes are

$$x_{12} = \sin \theta' \cos \pi \xi_4 = \sin \text{Arcsin } \sqrt{\xi_3} \cdot \cos \pi \xi_4 = \sqrt{\xi_3} \cos \pi \xi_4;$$
$$x_{22} = \sqrt{\xi_3} \sin \pi \xi_4;$$
$$x_{32} = \sqrt{1 - \xi_3}.$$

The new coordinate system (X_1', X_2', X_3') is obtained by rotation and parallel translation of the old axes. We find the coordinates $(x_{12}', x_{22}', x_{32}')$ in the old coordinate system. In matrix notation

$$(x_{12} x_{22} x_{32}) = (x_{12}' x_{22}' x_{32}') M + (x_{11} x_{21} x_{31}),$$

where M is the matrix of axis rotations

$$\begin{Vmatrix} -\xi_2 \cos \pi \xi_1; & \sin \pi \xi_1; & \sqrt{1 - \xi_2^2} \cos \pi \xi_1 \\ -\xi_2 \sin \pi \xi_1; & -\cos \pi \xi_1; & \sqrt{1 - \xi_2^2} \sin \pi \xi_1 \\ \sqrt{1 - \xi_2^2}; & 0; & \xi_2. \end{Vmatrix}$$

for which the elements of each column are the direction cosines of the new axes (X_1', X_2', X_3') in the old coordinate system X_1, X_2, X_3. The third column contains the components of the normal vector to the sphere at the given point. The zero asserts the condition of perpendicularity of the axes X_3 and X_2'. The remaining elements of M are obtained with allowance for the properties of the transformation determinant for rotation of the axes.

3. The point of intersection of the ray and a surface is found by solving the system of equations for the surface and the ray through the points (x_{11}, x_{21}, x_{31}) and (x_{12}, x_{22}, x_{32}):

$$\sum_{ij}^{4} A_{ij} x_i x_j = 0, \tag{4}$$

$$(j = 2, 3).$$

$$\frac{x_1 - x_{11}}{x_{12} - x_{11}} = \frac{x_j - x_{j1}}{x_{j2} - x_{j1}} \tag{5}$$

Solving the given system, we find the coordinates of the point of intersection (x_1, x_2, x_3).

It can happen that not every point of intersection of a surface and a random ray is situated in the investigated domain (i.e., the domain of existence of the surface).

The positions of the points are verified by means of check points, i.e., points that lie inside the sphere S and are such that only the domain of existence of one or more surfaces is visible from them; points lying outside the check-point domain, on the other hand, are not visible, i.e., they are shadowed. The surfaces in this case are considered to be nontransparent. For example, a suitable check point for a cube is a single point located inside the cube. All inner surfaces of the faces of the cube are visible from that point, whereas points situated on planes coplanar with, but exclusive of the face are not visible from the given check point. Consequently, to verify the position of the point of intersection of a ray with a surface it is required to ascertain whether that point is visible from a check point or whether it is shadowed from it. If it is visible from the check point, the random ray intersects the surface in the domain of existence of the latter, otherwise the point of intersection lies outside the investigated part of the surface.

TABLE 1. Generalized Geometrical Factors in Radiant Transfer for the Faces of a Cube

Relative position of faces	Average absorption coefficient of medium	Generalized geometrical factor		Number of elementary experiments
		from data of [3, 4]	maximum deviation by proposed technique	
Adjacent	0	0,200	0,055	110
			0,066	225
			0,021	767
			0,012	1148
			0,009	8161
			0,004	10561
			0,004	12105
			0,004	
	0,5	0,1539	0,004	12524
	2	0,0827	0,004	
Opposite	0	0,2000	0,003	
	0,5	0,1149	0,001	
	2	0,0221	0,0004	

For the verification procedure a straight line is drawn through the check point and the point of intersection with the ray, and the points of intersection of that line with the surfaces of the system are found; then it is determined whether some point of intersection lies between the check point and the point of intersection with the ray. Otherwise the logic involved is self-evident. If there are several check points (as in a complex system), the verification procedure is carried out from each check point. A point of intersection lies in the domain of existence if it is visible from at least one check point.

The execution of operations 4 and 5 of the algorithm does not require any explanation. The algorithm was implemented on a Minsk-22 digital computer. A comparison of the results of calculating the geometrical factors for the most typical cases of radiant heat transfer with the established data [4, 3] indicates satisfactory agreement at a reasonable computation time. Selected comparisons of the results from calculations of the geometrical factors for a cube whose interior is filled with an absorbing medium with the data of [1, 3] are made in Table 1.

Each of the deviations given in the table is the maximum of the deviations occurring for a particular face of the cube for a given number of experiments. The computation time for real systems of second-order surfaces (six to ten surfaces) with engineering accuracy is a few hours.

NOTATION

x_1, x_2, x_3	are the coordinates of a point;
l	is the length of a straight-line segment;
ρ, β, θ	are the radius, longitude, and polar distances, respectively, in spherical coordinates;
N	is the number of rays;
ξ	is a random number;
A_{ij}	represents the coefficients of the surface equation;
φ_{ij}	represents the generalized geometrical factors;
K	is the average absorption coefficient of the medium.

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